Applied General Topology

Appl. Gen. Topol. 21, no. 2 (2020), 295-304
doi:10.4995/agt.2020.13156

Rough action on topological rough groups

Alaa Altassan [a], Nof Alharbi [a], Hassen Aydi [b,c], Cenap Özel [a]

[a] Department of Mathematics, King Abdulaziz University, Jeddah, Kingdom of Saudi Arabia (aaltassan@kau.edu.sa,nof20081900@hotmail.com,cozel@kau.edu.sa)
[b] Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia. (hassen.aydi@isima.rnu.tn)
[c] China Medical University Hospital, China Medical University, Taichung 40402, Taiwan. (hassen.aydi@isima.rnu.tn)

Communicated by J. Galindo

Abstract

In this paper we explore the interrelations between rough set theory and group theory. To this end, we first define a topological rough group homomorphism and its kernel. Moreover, we introduce rough action and topological rough group homeomorphisms, providing several examples. Next, we combine these two notions in order to define topological rough homogeneous spaces, discussing results concerning open subsets in topological rough groups.

2010 MSC: *22A05; 54A05; 03E25.*

Keywords: *rough groups; topological rough groups; topological rough subgroups; product of topological rough groups; topological rough group homomorphisms; topological rough group homeomorphisms; topologically rough homogeneous spaces; rough kernel.*

1. Introduction

Rough Set Theory has many applications in economic, medicine and engeneering [13, 14, 16, 17]. Such a theory was introduced by Pawlak [21] and deals with uncertainty, impression and vagueness in information systems.The starting point of his analysis is the notion of *approximation space*, namely a pair(U, R), where U is any arbitrary non-empty set, called *universe*, and R is

Received 18 February 2020 – Accepted 28 August 2020

an equivalence relaon U. The set U/R of all equivalence classes $[x]_R$ forms a partition of U . Moreover, for any $X \subset U$, he introduced the notions of lower and upper approximations of X as follows:

$$\overline{X} = \{[x]_R : [x]_R \cap X \neq \varnothing\}$$

and

$$\underline{X} = \{[x]_R : [x]_R \subset X\}.$$

Next, he defined the rough set to be the orderd pair $X = (\underline{X}, \overline{X})$.

Recently, the interrelations between rough set theory and various branches of mathematics, such as combinatorics [12], monoids [10], matroids [15, 23, 24, 25], groups [7, 18], integral domains [11] and modules [9] has been deeply studied and constitute a research field which is developing rapidly. In our perspective, we are interested in the interrelations between rough set theory and groups. To this regard, let us first recall that in [7] and [18] the notions of rough groups, rough subgroups, rough homomorphisms and rough antihomomorphisms have been analyzed in detail. Moreover, the notion of topological rough groups was introduced by Bagirmaz et al in [6].

In this paper, we present rough actions and rough homogenous spaces, and discuss some of their properties. We also define a rough kernel. We organise the paper as follows. In section 2, we collect the needed material about rough groups and rough homomorphisms. Then the definition of topological rough groups and important properties have been recalled in section 3. Section 4 presents our main results where we introduce rough action and homogenous spaces.

This paper is produced from the PhD thesis of Ms. Nof Alharbi registered in King Abdulaziz University.

2. Rough groups and rough homomorphisms

In this section, we recall rough groups, rough homomorphisms and some of their properties.

Let (U, R) be an approximation space, where U is a non-empty set and R is an equivalence relation on U. Let $(*)$ be a binary operation defined on U. For all $x, y \in U$, we write xy instead of $x * y$. In 1994, Biswas and Nanda introduced the definition of rough groups which is given in the following definition.

Definition 2.1 ([6])**.** Let (U, R) be an approximation space. Suppose that G is a subset of U and $\overline{G}$ and $\underline{G}$ are respectively its upper and lower approximations. Then the rough set $G = (\underline{G}, \overline{G})$ is called a rough group if the following conditions are satisfied:

(1) $\forall x, y \in G, xy \in \overline{G}$.
(2) $(xy)z = x(yz), \forall x, y, z \in \overline{G}$.

(3) $\forall x \in G$, $\exists e \in \overline{G}$ such that $xe = ex = x$.
(4) $\forall x \in G, \exists y \in G$ such that $xy = yx = e$.

Definition 2.2 ([6]). A non-empty rough subset $H = (\underline{H}, \overline{H})$ of a rough group $G = (\underline{G}, \overline{G})$ is called a rough subgroup if it is a rough group itself.

Note that $G = (\underline{G}, \overline{G})$ is a trivial rough subgroup of itself. Moreover, if $e \in G$, then $e = (\underline{e}, \overline{e})$ is a trivial rough subgroup of the rough group G.

Theorem 2.3 ([6]). *Suppose that G is a subset of U and $\overline{G}$ and $\underline{G}$ are respectively its upper and lower approximations. Then a rough subset H is a rough subgroup of a rough group G if*
(1) $\forall x, y \in H, xy \in \overline{H}$;
(2) $\forall y \in H, y^{-1} \in H$.

Let H be a rough subgroup of a rough group G. Then H is said to be a rough normal subgroup of G if $xH = Hx, \forall x \in G$

Definition 2.4 ([18]). Let (U_1, R_1) and (U_2, R_2) be approximation spaces and $*, *^{'}$ be binary operations on U_1 and U_2, respectively. Let $G_1 \subseteq U_1$ and $G_2 \subseteq U_2$ be two rough groups. If the mapping $\varphi : \overline{G_1} \to \overline{G_2}$ satisfies that $\varphi(x * y) = \varphi(x) *^{'} \varphi(y)$, for all $x, y \in \overline{G_1}$, then φ is called **a rough homomorphism**.

Definition 2.5 ([18]). Let G_1 and G_2 be two rough groups. A rough homomorphism $\varphi : \overline{G_1} \to \overline{G_2}$ is said to be :
(1) a rough epimorphism (or surjective) if $\varphi : \overline{G_1} \to \overline{G_2}$ is onto.
(2) a rough monomorphism if $\varphi : \overline{G_1} \to \overline{G_2}$ is one-to-one.
(3) a rough isomorphism if $\varphi : \overline{G_1} \to \overline{G_2}$ is both onto and one-to-one.

Example 2.6. Let $(\mathbb{R}, R)$ be an approximation space, where $\mathbb{R}$ is the set of real numbers under addition. Consider the partition $\mathbb{R}/R = \{\mathbb{Q}, \mathbb{Q}^c\}$, where $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Q}^c$ is the set of irrational numbers. Let $G_1 = \mathbb{Q}$, and $G_2 = \mathbb{R}^* = \mathbb{R} - 0$. Then $\overline{G_1} = \mathbb{Q}$ and $\overline{G_2} = \mathbb{R}$. It is clear that G_1 and G_2 are rough groups. Define $\varphi : \mathbb{Q} \to \mathbb{R}$ as follow: for every $x \in \mathbb{Q}$, $\varphi(x) = x$. It is not difficult to see that φ is a rough monomorphism.

Example 2.7. Let $U = Z_4$ and consider the partition $U/R = \{\{\overline{1}, \overline{2}\}, \{\overline{0}, \overline{3}\}\}$. Let $G_1 = \{\overline{0}, \overline{2}\}$, and $G_2 = \{\overline{1}, \overline{2}, \overline{3}\}$. Then $\overline{G_1} = Z_4$, and $\overline{G_2} = Z_4$. It is clear that G_1 and G_2 are rough groups. Define $\varphi : \overline{G_1} \to \overline{G_2}$ as follows: $\varphi(x) = x$, $\forall x \in \overline{G_1}$. It is not difficult to see that φ is a rough epimorphism and a rough monomorphism. Thus φ is a rough isomorphism.

3. Topological rough groups

Throughout this section, we recall the definition of topological rough groups and we give some examples. For more details and properties of these structures, we refer the reader to [6].

Definition 3.1 ([6]). A rough group G with a topology τ on $\overline{G}$ is called a topological rough group if the following hold.

(1) $f : G \times G \to \overline{G}$ which defined by $f(x, y) = xy$ is continuous with respect to a product topology on $G \times G$ and the topology τ_G on G induced by τ;
(2) $\iota : G \to G$ which defined by $\iota(x) = x^{-1}$ is continuous with respect to the topology τ_G on G induced by τ.

Now, we present three different examples of topological rough groups.

Example 3.2. Let $U = Z_4$ be the group of integers modulo 4. Let $U/R = \{\{\overline{0}, \overline{2}, \overline{3}\}, \{\overline{1}\}\}$ be a classification of an equivalence relation and $G = \{\overline{1}, \overline{2}, \overline{3}\}$. Then $\underline{G} = \{\overline{1}\}$ and $\overline{G} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\} = Z_4$.
Given a topology $\tau = \{\varnothing, \overline{G}, \{\overline{1}\}, \{\overline{2}\}, \{\overline{3}\}, \{\overline{1}, \overline{2}\}, \{\overline{1}, \overline{3}\}, \{\overline{2}, \overline{3}\}, \{\overline{1}, \overline{2}, \overline{3}\}\}$ on $\overline{G}$. Then the relative topology on G is $\tau_G = \{\varnothing, G, \{\overline{1}\}, \{\overline{2}\}, \{\overline{3}\}, \{\overline{1}, \overline{2}\}, \{\overline{1}, \overline{3}\}, \{\overline{2}, \overline{3}\}\}$.
The two conditions in Definition 3.1 are satisfied as follows:

(1) The product mapping $f : G \times G \to \overline{G} = Z_4$ is continuous with respect to product topology on $G \times G$ and the topology τ_G on G induced by the topology τ on $\overline{G} = Z_4$. For instance, the open set $\{\overline{1}, \overline{2}\}$ in τ_G has inverse $\{\{\overline{3}\} \times \{\overline{2}\} \cup \{\overline{3}\} \times \{\overline{3}\}\}$ which is open in the product topology.
(2) The inverse mapping $i : G \to G$ is continuous with respect to the topology τ_G on G induced by the topology τ. For instance the open set $\{\overline{1}\}$ has inverse $\{\overline{3}\}$ which is open in τ_G.

Hence G is a topological rough group.

Example 3.3. Let $U = \mathbb{R}$ and $U/R = \{\{x : x \geq 0\}, \{x : x < 0\}\}$ be a partition of U. Consider $G = \mathbb{R}^* = \mathbb{R} - 0$. Then $\underline{G} = \{x : x < 0\}$, $\overline{G} = \mathbb{R}$. And G is a rough additive group. Let $\mathcal{D}$ be the discrete topology on $\overline{G} = \mathbb{R}$, then

(1) the product mapping $f : \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}$ is continuous with respect to product topology on $\mathbb{R}^* \times \mathbb{R}^*$ and the topology $\mathcal{D}_G$ on $\mathbb{R}^*$ induced by the discrete topology $\mathcal{D}$ on $\mathbb{R}$.
(2) The inverse mapping $i : \mathbb{R}^* \to \mathbb{R}^*$ is continuous with respect to topology $\mathcal{D}_{\mathcal{G}}$ on $\mathbb{R}^*$ induced by the discrete topology $\mathcal{D}$.

Therefore the rough group G is a topological rough group with the discrete topology $\mathcal{D}$ on $\overline{G} = \mathbb{R}$.

Example 3.4 ([6])**.** Let $U = S_4$ be the set of all permutations of four objects and $(*)$ be the multiplication operation of permutations. Consider

$$U/R = \{E_1, E_2, E_3, E_4\},$$

to be a partition of U, where

$$E_1 = \{1, (12), (13), (14), (23), (24), (34)\}$$
$$E_2 = \{(123), (132), (142), (124), (134), (143), (234), (243)\}$$
$$E_3 = \{(1234), (1243), (1342), (1324), (1423), (1432)\}$$
$$E_4 = \{(12)(34), (13)(24), (14)(23)\}.$$

For $G = \{(12), (123), (132)\}$, $\overline{G} = E_1 \cup E_2$. It is not difficult to see that G is a rough group.

For a given topology $\tau = \{\varnothing, \overline{G}, \{(12)\}, \{1, (132), (123)\}, \{1, (12), (132), (123)\}\}$ on $\overline{G}$, the relative topology on G is $\tau_G = \{\varnothing, G, \{(12)\}, \{(132), (123)\}\}$. Moreover by examine the two conditions in Definition 3.1, we can see that G is a topological rough group.

Definition 3.5 ([6])**.** Let G be a topological rough group. For a fixed element a in G, we define the following:

(1) A mapping $L_a : G \to \overline{G}$ which is defined by $L_a(x) = ax$, is called a left transformation from G into $\overline{G}$.
(2) A mapping $R_a : G \to \overline{G}$ which is defined by $R_a(x) = xa$, is called a right transformation from G into $\overline{G}$.

Proposition 3.6 ([6])**.** *Let G be a topological rough group. Then*

(1) *The left transformation map $L_a : G \to \overline{G}$ is continuous and one-to-one.*
(2) *The right transformation map $R_a : G \to \overline{G}$ is continuous and one-to-one.*
(3) *The inverse mapping $\iota : G \to G$ is a homeomorphism for all $x \in G$.*

4. Rough action and rough homogenous spaces in classical set topology

In this section, we discuss our main results. We introduce rough action and rough homogenous spaces in classical set topology using rough groups.

First, we recall cartesian product of topological rough groups. Let (U, R_1) and (V, R_2) be approximation spaces with binary operations $*_1$ and $*_2$, respectively. For $x, x^{'} \in U$ and $y, y^{'} \in V$, we have $(x, y), (x^{'}, y^{'}) \in U \times V$. Define $*$ as $(x, y) * (x^{'}, y^{'}) = (x *_1 x^{'}, y *_2 y^{'})$. Then $*$ is a binary operation on $U \times V$. In deed, that the product of equivalence relations R_1 and R_2 is also an equivalence relation on $U \times V$ (see [3]). Moreover, we have the following result.

Theorem 4.1 ([4])**.** *Let $G_1 \subseteq U$ and $G_2 \subseteq V$ be two rough groups. Then the cartesian product $G_1 \times G_2$ is also a rough group.*

Now, let (U, R_1) and (V, R_2) be approximation spaces. Let $G_1 \subseteq U$ and $G_2 \subseteq V$ be topological rough groups such that τ_1 and τ_2 are topologies on $\overline{G_1}$ and $\overline{G_2}$, respectively inducing τ_{G_1} and τ_{G_2} on G_1 and G_2, respectively.

A mapping $\varphi : \overline{G_1} \to \overline{G_2}$ is called **a topological rough group homomorphism**, if φ is a rough homomorphism and continuous with respect to the topology τ_2 on $\overline{G_2}$ inducing τ_{G_2} on G_2 and the topology τ_1 on $\overline{G_1}$ inducing τ_{G_1} on G_1.

A topological rough group homomorphism $\varphi : \overline{G_1} \to \overline{G_2}$ is called **a topological rough group homeomorphism**, if there exists a topological rough group homomorphism $\varphi^{-1} : \overline{G_2} \to \overline{G_1}$ such that $\varphi^{-1} \circ \varphi = 1_{\overline{G_1}}$.

The next definition is equivalent to the definition of rough kernel in rough groups that is given in [18].

Definition 4.2. Let G_1 and G_2 be topological rough groups, $\varphi : \overline{G_1} \to \overline{G_2}$ be a topological rough group homomorphism and let e_2 be the rough identity element in G_2. Then

$$ker(\varphi) = \{g \in G_1 : \varphi(g) = e_2\}.$$

is called the **rough kernel** associated with the map φ.

In the next theorem, we prove that, the kernel in Definition 4.2 is a rough normal subgroup of G_1.

Theorem 4.3. *Let φ be a topological rough group homomorphism from $\overline{G_1}$ to $\overline{G_2}$. Then the rough kernel is a rough normal subgroup of G_1.*

Proof. For every $x, y \in ker(\varphi)$, we have $\varphi(x) = e_2$, and $\varphi(y) = e_2$.

1. Since $\varphi(x * y) = \varphi(x) *^{'} \varphi(y) = e_2$, we have $x * y \in ker(\varphi)$.
2. We have $\varphi(x^{-1}) = (\varphi(x))^{-1} = (e_2)^{-1}$. Hence $ker(\varphi)$ is a rough subgroup of G_1.
3. For every $x \in G_1$ and $r \in ker(\varphi)$, we have $\varphi(x * r * x^{-1}) = \varphi(x) *^{'} \varphi(r) *^{'} \varphi(x^{-1}) = e_2$. Therefore, $x * r * x^{-1} \in ker(\varphi)$. Thus $ker(\varphi)$ is a rough normal subgroup of G_1.

□

Note that, the rough kernel is always a subset of the upper approximation of G_1. Indeed, if $\overline{G_1}$ is a group then the kernel is a normal subgroup of $\overline{G_1}$.

Example 4.4. Consider the map $\varphi : Z_4 \to \mathbb{R}$, where $G = \{\overline{1}, \overline{2}, \overline{3}\}$ and $\mathbb{R}^*$ are the rough groups in Example 3.2 and Example 3.3, respectively. Define φ as follows:

$$\varphi(\overline{0}) = 0, \varphi(\overline{1}) = 0, \varphi(\overline{2}) = 0, \varphi(\overline{3}) = 0.$$

Clearly, φ is continuous and homomorphism. Hence φ is a topological rough group homomorphism. From Definition 4.2, it is easy to see that $ker(\varphi) = \{\overline{1}, \overline{2}, \overline{3}\}$ is a subset of G. Moreover, $ker(\varphi)$ is a rough normal subgroup of G.

Let (U, R) be an approximation space. Assume moreover that G is a topological rough group and X is a subset of U. Denote by $X^{\dagger}$ the topological space inducing the topological space X; where X is a rough set with ordinary topology.

Now, we are ready to give the definition of the action of a rough group G on a rough space.

Definition 4.5. A continuous map $\varphi : \overline{G} \times X^{\dagger} \to X^{\dagger}$ (resp. $\varphi : X^{\dagger} \times \overline{G} \to X^{\dagger}$) is called a left (resp. right) rough action of G on X, if it satisfies the following conditions:

1. $g(g^{'}x) = (gg^{'})x$ (resp. $((xg)g^{'} = x(gg^{'}))$, for all $g, g^{'} \in \overline{G}$ and $x \in X^{\dagger}$.
2. $ex = x$ (resp. $xe = x$), for every $x \in X^{\dagger}$, where $e \in \overline{G}$ is the rough identity.

Then the rough set X is called a rough G-space.

The action φ is said to be **effective** if $gx = g^{'}x$, for every $x \in X^{\dagger}$ implies $g = g^{'}$. In addition, the action φ is said to be **transitive**, if for every $x, x^{'} \in X^{\dagger}$, there exists $g \in \overline{G}$ such that $gx = x^{'}$.

Definition 4.6. Let X be a rough G-space. Then X is said to be **topologically rough homogeneous** if for all $x, y \in X^{\dagger}$, there is a topological homeomorphism $\varphi : X^{\dagger} \to X^{\dagger}$ such that $\varphi(x) = y$.

The action of a topological rough group on itself is discussed in the following proposition.

Proposition 4.7. *Let H be a rough subgroup of the topological rough group G, and let $\overline{G}$ be a group. Then H acts on G. Moreover, G acts roughly on itself.*

Proof. Since H is a rough subgroup of G, $\overline{H}$ is a subset of the group $\overline{G}$. Therefore the continuous map $\varphi : \overline{H} \times \overline{G} \to \overline{G}$ is a left rough action of H on G. Also since $\overline{G}$ is a group, the continuous map $\varphi : \overline{G} \times \overline{G} \to \overline{G}$ is a left rough action of G on G. □

Theorem 4.8. *Let G be a topological rough group and X be a rough G- space. Then for every $g \in G$, the left transformation map $L_g : X^{\dagger} \to X^{\dagger}$, (resp. right transformation map $R_g : X^{\dagger} \to X^{\dagger}$) which is defined by $L_g(x) = gx$ ($R_g(x) = xg$), is a topological homeomorphism.*

Proof. Indeed, the continuity of the action φ implies the continuity of L_g. The conditions 1 and 2 in Definition 4.5 are respectively equivalent to

(1) $L_g \circ L_{g'} = L_{gg'}$.
(2) $L_e = 1_X$.

Therefore, the maps L_g and $L_{g^{-1}}$ are inverses of each other. Thus, L_g is a topological homeomorphism from $\overline{X}$ to $X^{\dagger}$. □

Note that, the left (resp. right) transformation map $L_g(R_g)$ from $X^{\dagger}$ into $X^{\dagger}$, is not in general a topological homeomorphism for every $g \in \overline{G}$. Indeed, this is only true in the case where $\overline{G}$ is a group.

From now on, we will focus on studying open subsets in topological rough groups.

Corollary 4.9. *Let G be topological rough group. Then for every open set O in $X^{\dagger}$ and $g \in G$, $L_g(O) = gO$ is open in $X^{\dagger}$.*

Proof. By Theorem 4.8, $L_g(O) = X^{\dagger} \to X^{\dagger}$ is a topological homeomorphism. Thus gO is open set in $X^{\dagger}$.

□

Theorem 4.10. *Let G be a topological rough group such that $\overline{G}$ is a group. For any open subset O of $\overline{G}$, if A is a subset of $\overline{G}$, then AO (respectively OA) is open in $\overline{G}$.*

Proof. The fact that $\overline{G}$ is a group implies that G acts on itself. Thus for every $g \in \overline{G}, L_g$ is a topological homeomorphism. The rest of proof follows immediately from left transformation definition. Therefore $AO = \cup_{a\in A} L_a(O)$. Similarly $OA = \cup_{a\in A} R_a(O)$ is open in $\overline{G}$. □

Theorem 4.11. *Let G be a topological rough group such that $\overline{G}$ is a group. Let H be a rough subgroup of G such that $\overline{H}$ is closed under multiplication. If there is an open set O in G such that $e \in O$ and $O \subseteq H$, then $\overline{H}$ is open set in $\overline{G}$.*

Proof. Let O be a non-empty open set in G such that $O \subseteq H$ and $e \in O$. Then for every $h \in \overline{H}$, $L_h(O) = hO$ is open in $\overline{G}$. Hence $\overline{H} = \bigcup_{h\in\overline{H}} hO$ is open in $\overline{G}$. □

Theorem 4.12. *Let G be a topological rough group such that $\overline{G}$ is a group and let H be a rough subgroup of G. Let O be an open set in G such that $O \subseteq H$. Then for every $h \in H$, hO is an open set in $\overline{H}$.*

Proof. Since $\overline{H} \subseteq \overline{G}$, and $\overline{G}$ is a group, L_h is a topological homeomorphism. By the definition of left transformation, $L_h(O) = hO$ is open in $\overline{G}$. The fact that $O \subseteq H$ implies $hO \subseteq \overline{H}$. Hence, hO is open in $\overline{H}$. □

Using the notion of open subsets in topological rough groups, we define the following set.

Definition 4.13. Let G be a topological rough group and let $\mathcal{B} \subseteq \tau$ be a base for τ. For $g \in G$, the family

$$\mathcal{B}_g = \{O \cap G : O \in \mathcal{B}, g \in O\} \subseteq \mathcal{B}$$

is called a **base at** g in τ_G.

Example 4.14. In Example 3.3 the family $\mathcal{B} = \{\{x\} : x \in \mathbb{R}\}$ is a base for $\mathcal{D}$. For every $g \in G$ the collection $\mathcal{B} = \{\{g\} : g \in \mathcal{R}^*\}$ is a base for τ_G.

Theorem 4.15. *Let G be a topological rough group such that the identity element $e \in G$ and $\overline{G}$ is closed under multiplication. Let G be an open set in $\overline{G}$. For $g \in G$ the base of g in $\overline{G}$ is equal to*

$$\mathcal{B}_g = \{gO : O \in \mathcal{B}_e\},$$

where $\mathcal{B}_e$ is the base of the identity e in τ_G.

Proof. Since $g \in G$, we have $g \in \overline{G}$. Let O_1 be an open set in $\overline{G}$ and let $g \in U$. Since $e \in G$, and G is a topological rough group, there are two open sets O_2 and O_3 such that $g \in O_2, e \in O_3$ and $\varphi(O_2 \times O_3) \subseteq O_1$. We have G is an open set in τ. Then O_3 is a neighbourhood of e in τ. Then there is a basic open set $O \in \mathcal{B}_e$ such that $e \in O \subseteq O_3$. Hence $L_g(O) = gO \subseteq \varphi(O_2 \times O) \subseteq \varphi(O_2 \times O_3) \subseteq O_1$. □

Acknowledgement The authors wish to thank the Deanship for Scientific Research (DSR) at King Abdulaziz University for financially funding this project under grant no. KEP-PhD-2-130-39. Also, we would like to thank the editor and referees for their valuable suggestions which have improved the presentation of the paper.

References

[1] S. Akduman, E. Zeliha, A. Zemci and S. Narli, Rough topology on covering based rough sets, 1st International Eurasian Conference on Mathematical Sciences and Applications (IECMSA), Prishtine, Kosovo, 3ÔÇô7 September 2012.

[2] S. Akduman, A. Zemci and C. Özel, Rough topology on covering-based rough sets, Int. J. Computational Systems Engineering 2, no. 2 (2015),107–111.

[3] N. Alharbi, H. Aydi and C. Özel, Rough spaces on covering based rough sets, European Journal of Pure And Applied Mathematics (EJPAM) 12, no. 2 (2019).

[4] N. Alharbi, H. Aydi, C. Park and C. Özel, On topological rough groups, J. Computational Analysis and Applications 29, no. 1 (2021), 117 –122.

[5] A. Arhangel'skii and M. Tkachenko, Topological groups and related structures, Atlantis press/ World Scientific, Amsterdam-Paris, 2008.

[6] N. Bagirmaz, I. Icen and A. F. Ozcan, Topological rough groups, Topol. Algebra Appl. 4 (2016), 31–38.

[7] R. Biswas and S. Nanda, Rough groups and rough subgroups, Bull. Polish Acad. Sci. Math. 42 (1994), 251–254.

[8] E. Brynairski, A calculus of rough sets of the first order, Bull. of the Polish Academy Sciences: Mathematics 37, no. 1-6 (1989), 71–78.

[9] G. Chiaselotti and F. Infusino, Some classes of abstract simplicial complexes motivated by module theory, Journal of Pure and Applied Algebra 225 (2020), 106471,

[10] G. Chiaselotti and F. Infusino, Alexandroff topologies and monoid actions, Forum Mathematicum 32, no. 3 (2020), 795–826.

[11] G. Chiaselotti, F. Infusino and P. A. Oliverio, Set relations and set systems induced by some families of integral domains, Advances in Mathematics 363 (2020), 106999,

[12] G. Chiaselotti, T. Gentile and F. Infusino, Lattice representation with algebraic granular computing methods, Electronic Journal of Combinatorics 27, no. 1 (2020), P1.19.

[13] S. Hallan, A. Asberg and T. H. Edna, Additional value of biochemical tests in suspected acute appendicitis, European Journal of Surgery 163, no. 7 (1997), 533–538.

[14] R. R. Hashemi, F. R. Jelovsek and M. Razzaghi, Developmental toxicity risk assessment: A rough sets approach, Methods of Information in Medicine 32, no. 1 (1993), 47–54.

[15] A. Huang, H. Zhao and W. Zhu, Nullity-based matroid of rough sets and its application to attribute reduction, Information Sciences 263 (2014), 153–165.

[16] A. Kusiak, Decomposition in data mining: An industrial case study, IEEE Transactions on Electronics Packaging Manufacturing 23 (2000), 345–353.

[17] A. Kusiak, Rough set theory: A data mining tool for semiconductor manufacturing, IEEE Transactions on Electronics Packaging Manufacturing 24, no. 1(2001), 44–50.

[18] C. A. Neelima and P. Isaac, Rough anti-homomorphism on a rough group, Global Journal of Mathematical Sciences: Theory and Practical 6, no. 2, (2014), 79–80.

[19] M. Novotny and Z. Pawlak, On rough equalities, Bulletin of the Polish Academy of Sciences, Mathematics 33, no. 1-2 (1985), 99–104.

[20] N. Paul, Decision making in an information system via new topology, Annals of fuzzy Mathematics and Informatics 12, no. 5 (2016), 591–600.

[21] Z. Pawlak,Rough sets, Int. J. Comput. Inform. Sci. 11, no. 5 (1982), 341–356.

[22] J. Pomykala, The stone algebra of rough sets, Bulletin of the Polish Academy of Sciences, Mathematics 36, no. 7-8 (1988), 495–508.

[23] J. Tanga, K. Shea, F. Min and W. Zhu, A matroidal approach to rough set theory, Theoretical Computer Science 471 (2013), 1–11.

[24] S. Wang, Q. Zhu, W. Zhu and F. Min, Graph and matrix approaches to rough sets through matroids, Information Sciences 288 (2014), 1–11.

[25] S. Wang, Q. Zhu, W. Zhu and F. Min, Rough set characterization for 2-circuit matroid, Fundamenta Informaticae 129 (2014), 377–393.