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Simple polynomial equations over $(m \times m)$-matrices

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Abstract

Let m be any integer ≥ 3. We consider the polynomial equation

$$X^n + a_{n-1} \cdot X^{n-1} + \cdots + a_1 \cdot X + a_0 \cdot I = O,$$

over $(m \times m)$-matrices X with the real entries, where I is the identity matrix, O is the null matrix, $a_i \in \mathbb{R}$ for each i and $n \geq 1$. We discuss its solution set S supplied with the natural Euclidean topology. In particular, we describe the solution set S for $m = 3$ and calculate its dimension.

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1. Introduction

Let m be any integer ≥ 2. As usual, by $M_m(\mathbb{R})$ we denote the algebra of real $(m \times m)$-matrices, and by $GL_m(\mathbb{R})$ the group of invertible $(m \times m)$-matrices. We equip $M_m(\mathbb{R})$ with any of equivalent norms that make $M_m(\mathbb{R})$ a topological algebra, homeomorphic to $\mathbb{R}^{m^2}$.

Consider a polynomial

$$F(X) = X^n + A_{n-1}X^{n-1} + \cdots + A_1X + A_0$$

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of degree $n \geq 2$, where X and A_i, $i = 0, 1, \dots, n-1$, are from $M_m(\mathbb{R})$, and the equation

$$F(X) = O \qquad (1),$$

where O is the null $(m \times m)$-matrix.

Any matrix $B \in M_m(\mathbb{R})$ such that $F(B) = O$ is called *a solution* of (1). *The solution set* of (1) consists of all solutions of (1).

In [8] Wilson demonstrated a method showing that the solution set of (1) with $n = 2$ over $M_2(\mathbb{R})$ can consist of exactly $0, 1, 2, 3, 4, 5, 6$ elements or it can be infinite (see also [4] and [7]). The method uses eigenvalues and eigenvectors and is due to Fuchs and Schwarz [3].

One can simplify the polynomial $F(X)$ by setting $A_k = a_k \cdot I$ for each $0 \leq k \leq n-1$, where each a_k is a real number. Denote the polynomial $F(X)$ in this case by $F_s(X)$ and note that

$$F_s(X) = X^n + a_{n-1} \cdot X^{n-1} + \cdots + a_1 \cdot X + a_0 \cdot I.$$

Continue with the equation

$$F_s(X) = O \qquad (2).$$

Here we consider (2) for all $m \geq 3$. (The case $m = 2$ was described in [1].)

Everywhere in this note, by a space we mean a metric separable space. The covering dimension of a space X will be denoted by $\dim X$. See [2, p. 385] for the definition of dim and its main properties, such as:

(i) *The countable sum theorem* [2, Theorem 7.2.1]: if a space X is the union $\cup_{i=1}^{\infty} X_i$, where each X_i is a closed subset of X with $\dim X_i \leq n, n \geq 0$, then $\dim X \leq n$, and

(ii) *The monotone theorem* [2, Theorem 7.3.4]: for every subspace A of a space X we have $\dim A \leq \dim X$.

Recall that $\dim \mathbb{R}^k = k$ and for any non-empty space X of cardinality less than continuum we have $\dim X = 0$.

It is easy to see that the solution set S of (2) (and of (1) as well) is a closed subset of $M_m(\mathbb{R})$ and hence $\dim S \leq m^2$. In fact, we can easily improve this estimate (similarly to [1, Proposition 1.2]) as follows.

Proposition 1.1. $\dim S \leq m^2 - 1$.

(The same is valid for the solution set of (1)*.)*

In [1, Theorem 3.4] we described the solution set of (2) for $m = 2$ and showed (see [1, Theorem 4.5]) that its dimension is equal to 2 for all $n \geq 2$.

Thus the following question appears naturally.

Question 1.2 ([1, Question 7.2])**.** *Let $m \geq 3$. What is the structure of S? What is the dimension of S?*

2. Preliminaries I

For a given $B \in M_m(\mathbb{R})$, define a map

$$Conj_B \colon GL_m(\mathbb{R}) \to M_m(\mathbb{R})$$

by $Conj_B(X) = X^{-1}BX$.

The following is easy to verify.

Lemma 2.1. *The map $Conj_B$ is continuous, and the image $Conj_B(GL_m(\mathbb{R}))$ is σ-compact.*

For a given closed subgroup G of $GL_m(\mathbb{R})$ denote by $GL_m(\mathbb{R})/G$ the quotient space of the right cosets of G in $GL_m(\mathbb{R})$, and by $\pi : GL_m(\mathbb{R}) \to GL_m(\mathbb{R})/G$ the canonical mapping.

Since the topological group $GL_m(\mathbb{R})$ is locally compact, by the use of a well known fact (see, for example, [6, Theorem 2]) we have the following equality:

$$\dim GL_m(\mathbb{R}) = \dim G + \dim GL_m(\mathbb{R})/G.$$

For a given $B \in M_m(\mathbb{R})$ let $G_B = \{A \in GL_m(\mathbb{R}) : AB = BA\}$. It is easy to see that G_B is a closed subgroup of $GL_m(\mathbb{R})$.

The following is evident.

Proposition 2.2.

(i) *Let $X, Y \in GL_m(\mathbb{R})$. If $Conj_B(X) = Conj_B(Y)$, then $Y = AX$ for some $A \in G_B$.*
(ii) *For any $X \in GL_m(\mathbb{R})$ and $A \in G_B$ we have $Conj_B(X) = Conj_B(AX)$.*
(iii) *There exists a continuous bijection $\alpha_B : GL_m(\mathbb{R})/G_B \to Conj_B(GL_m(\mathbb{R}))$ such that $Conj_B = \alpha_B\pi$.*

Since the space $GL_m(\mathbb{R})/G_B$ is σ-compact and normal (see [6, Theorem 1]), we get the following corollary.

Corollary 2.3. $\dim Conj_B(GL_m(\mathbb{R})) = \dim GL_m(\mathbb{R})/G_B = \dim GL_m(\mathbb{R}) - \dim G_B$.

Recall that $(m \times m)$-matrices B_1 and B_2 with entries from a field $\mathbb{F}$ are *similar* over $\mathbb{F}$ if there exists an invertible $(m \times m)$-matrix C with entries from $\mathbb{F}$ such that $B_1 = C^{-1}B_2C$, and the similarity is an equivalence relation.

The following is evident.

Proposition 2.4. *For any matrices B_1 and B_2 from $M_m(\mathbb{R})$ we have the sets $Conj_{B_1}(GL_m(\mathbb{R}))$ and $Conj_{B_2}(GL_m(\mathbb{R}))$ coincide or they are disjoint.*

In particular, for any $B \in M_m(\mathbb{R})$ we have that $Conj_B(GL_m(\mathbb{R})) = Conj_J(GL_m(\mathbb{R}))$, where J is a real Jordan canonical form of B.

3. Preliminaries II

For distinct real numbers a, b, c and real numbers p, q, where $q \neq 0$, let

$$J(a,b,c) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},\ J(a,a,b) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix},$$

$$J_1(a,a,b) = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix},\ J(a,a,a) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix},$$

$$J_1(a,a,a) = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix},\ J_2(a,a,a) = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix},$$

$$J_c(a,p,q) = \begin{pmatrix} a & 0 & 0 \\ 0 & p & q \\ 0 & -q & p \end{pmatrix},\ U(a,p,q) = \begin{pmatrix} a & 0 & 0 \\ 0 & p+iq & 0 \\ 0 & 0 & p-iq \end{pmatrix}.$$

The following is easy to verify.

Proposition 3.1.

(i) $G_{J(a,b,c)} = \{A = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} : \det A \neq 0\}$,
in particular, $\dim G_{J(a,b,c)} = 3$.

(ii) $G_{J(a,a,b)} = \{A = \begin{pmatrix} x & y & 0 \\ z & t & 0 \\ 0 & 0 & s \end{pmatrix} : \det A \neq 0\}$,
in particular, $\dim G_{J(a,a,b)} = 5$.

(iii) $G_{J_1(a,a,b)} = \{A = \begin{pmatrix} x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & z \end{pmatrix} : \det A \neq 0\}$,
in particular, $\dim G_{J_1(a,a,b)} = 3$.

(iv) $G_{J(a,a,a)} = GL_3(\mathbb{R})$, *in particular,* $\dim G_{J(a,a,a)} = 9$.

(v) $G_{J_1(a,a,a)} = \{A = \begin{pmatrix} x & y & z \\ 0 & x & 0 \\ 0 & s & t \end{pmatrix} : \det A \neq 0\}$,
in particular, $\dim G_{J_1(a,a,a)} = 5$.

(vi) $G_{J_2(a,a,a)} = \{A = \begin{pmatrix} x & y & z \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} : \det A \neq 0\}$,
in particular, $\dim G_{J_2(a,a,a)} = 3$.

(vii) $G_{J_c(a,p,q)} = \{A = \begin{pmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & -z & y \end{pmatrix} : \det A \neq 0\}$,
in particular, $\dim G_{J_c(a,p,q)} = 3$.

The following propositions are standard facts from matrix algebra (see, for example, [5]).

Proposition 3.2. *For any matrix $B \in M_3(\mathbb{R})$ there exists a matrix $C \in GL_3(\mathbb{R})$ such that $B = C^{-1}JC$, where J is precisely one of the following matrices:*
$J(a,b,c), J(a,a,b), J_1(a,a,b), J(a,a,a), J_1(a,a,a), J_2(a,a,a), J_c(a,p,q)$.

Proposition 3.3. *Any $B \in M_3(\mathbb{R})$ is similar to $J_c(a,p,q)$ over $\mathbb{R}$ iff B is similar to $U(a,p,q)$ over $\mathbb{C}$.*

4. The structure and the dimension of S, the case $m = 3$.

In this section we completely describe the structure of the solution set S and compute its dimension when $m = 3$.

Let $f_s(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with real coefficients $a_i, 0 \leq i \leq n-1$, corresponding to the polynomial $F_s(X)$.

Consider the equation

$$f_s(x) = 0 \qquad (3).$$

The following lemmas are evident.

Lemma 4.1.

$$F_s(J(a,b,c)) = \begin{pmatrix} f_s(a) & 0 & 0 \\ 0 & f_s(b) & 0 \\ 0 & 0 & f_s(c) \end{pmatrix};\ F_s(J(a,a,b)) = \begin{pmatrix} f_s(a) & 0 & 0 \\ 0 & f_s(a) & 0 \\ 0 & 0 & f_s(b) \end{pmatrix};$$

$$F_s(J_1(a,a,b)) = \begin{pmatrix} f_s(a) & f_s'(a) & 0 \\ 0 & f_s(a) & 0 \\ 0 & 0 & f_s(b) \end{pmatrix};\ F_s(J(a,a,a)) = \begin{pmatrix} f_s(a) & 0 & 0 \\ 0 & f_s(a) & 0 \\ 0 & 0 & f_s(a) \end{pmatrix};$$

$$F_s(J_1(a,a,a)) = \begin{pmatrix} f_s(a) & f_s'(a) & 0 \\ 0 & f_s(a) & 0 \\ 0 & 0 & f_s(a) \end{pmatrix};\ F_s(J_2(a,a,a)) = \begin{pmatrix} f_s(a) & f_s'(a) & \frac{1}{2}f_s''(a) \\ 0 & f_s(a) & f_s'(a) \\ 0 & 0 & f_s(a) \end{pmatrix};$$

$$F_s(U(a,p,q)) = \begin{pmatrix} f_s(a) & 0 & 0 \\ 0 & f_s(p+iq) & 0 \\ 0 & 0 & f_s(p-iq) \end{pmatrix}.$$

In particular,
a, b, c are distinct real roots of (3) iff $J(a,b,c)$ is a solution of (2);
a, b are distinct real roots of (3) iff $J(a,a,b)$ is a solution of (2);
a, b are distinct real roots of (3) and a is a root of multiplicity ≥ 2 iff $J_1(a,a,b)$ is a solution of (2);
a is a real root of (3) iff $J(a,a,a)$ is a solution of (2);
a is a real root of (3) of multiplicity ≥ 2 iff $J_1(a,a,a)$ is a solution of (2);
a is a real root of (3) of multiplicity ≥ 3 iff $J_2(a,a,a)$ is a solution of (2);
a is a real root of (3) and $p \pm iq$ are non-real roots of (3) iff $F_s(U(a,p,q)) = O$.

Lemma 4.2. *If $(m \times m)$-matrices A and B (with real or complex entries) are similar over $\mathbb{R}$ or $\mathbb{C}$, and $F_s(A) = O$, then $F_s(B) = O$.*

In particular, $J_c(a,p,q)$ is a solution of (2) iff a is a real root of (3) and $p \pm iq$ are non-real roots of (3).

Let $S(B) = Conj_B(GL_3(\mathbb{R}))$ for any $B \in M_3(\mathbb{R})$.

Using Propositions 3.2, 3.3, and Lemmas 4.1, 4.2 we get the following theorem.

Theorem 4.3. *The solution set S of the equation (2), $m = 3$, is either empty or a disjoint union of the following sets:*

$S(J(a,b,c))$, whenever a, b, c are distinct real roots of (3),
$S(J(a,a,b))$, whenever a, b are distinct real roots of (3),

$S(J_1(a, a, b))$, *whenever* a, b *are distinct real roots of* (3) *and* a *is a root of multiplicity* ≥ 2,
$S(J(a, a, a))$, *whenever* a *is a real root of* (3),
$S(J_1(a, a, a))$, *whenever* a *is a real root of* (3) *of multiplicity* ≥ 2,
$S(J_2(a, a, a))$, *whenever* a *is a real root of* (3) *of multiplicity* ≥ 3,
$S(J_c(a, p, q))$, $q > 0$, *whenever* a *is a real root of* (3) *and* $p \pm iq$ *are non-real roots of* (3).

Using Proposition 3.1 and Corollary 2.3 we get the following corollary.

Corollary 4.4.

(i) $\dim S(J(a, b, c)) = \dim S(J_1(a, a, b)) =$
$\dim S(J_2(a, a, a)) = \dim S(J_c(a, p, q)) = 6.$

(ii) $\dim S(J(a, a, b)) = \dim S(J_1(a, a, a)) = 4.$

(iii) $\dim S(J(a, a, a)) = 0$, *moreover,*

$$S(J(a, a, a)) = \{\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}\}.$$

Corollary 4.5. *Let* $m = 3$. *Then the solution set* S *of the equation* (2) *is* σ*-compact, and* $\dim S$ *is equal to* -1 *or it is the maximum of dimensions of sets* $S(J)$ *from Theorem 4.3, i.e.* $0, 4$ *or* 6.

5. Some examples of solution sets, the case $m = 3$.

Here we will describe the solution sets of some equations. All facts here follow from the results of Section 4.

Example 5.1. The solution set S_1 of $X^2 + I = O$ is empty, and so $\dim S_1 = -1$.

Example 5.2.

(i) The solution set S_2 of $X = O$ is equal to $S(J(0, 0, 0)) = \{O\}$, and so $\dim S_2 = 0$.

(ii) The solution set S_3 of $X - I = O$ is equal to $S(J(1, 1, 1)) = \{I\}$, and so $\dim S_3 = 0$.

Example 5.3. The solution set S_4 of $X^2 = O$ is equal to $S(J(0, 0, 0)) \cup S(J_1(0, 0, 0))$, and so $\dim S_4 = 4$.

Example 5.4. The solution set S_5 of $X^2(X-I) = O$ is equal to $S(J(0, 0, 0)) \cup S(J_1(0, 0, 0)) \cup S(J(1, 1, 1)) \cup S(J(0, 0, 1)) \cup S(J(1, 1, 0)) \cup S(J_1(0, 0, 1))$, and so $\dim S_5 = 6$.

Moreover, $S_5 \setminus (S_3 \cup S_4) = S(J(0, 0, 1)) \cup S(J(1, 1, 0)) \cup S(J_1(0, 0, 1))$,
and so $\dim S_5 \setminus (S_3 \cup S_4) = 6 > 4 = \dim(S_3 \cup S_4)$.

Example 5.5. The solution set S_6 of $X^2 - I = O$ is equal to $S(J(1, 1, 1)) \cup S(J(-1, -1, -1)) \cup S(J(1, 1, -1)) \cup S(J(-1, -1, 1))$, and so $\dim S_6 = 4$.

Example 5.6. The solution set S_7 of $X(X^2-I) = O$ is equal to $S(J(0,0,0)) \cup S(J(1,1,1)) \cup S(J(-1,-1,-1)) \cup S(J(0,0,1)) \cup S(J(1,1,0)) \cup S(J(0,0,-1)) \cup S(J(-1,-1,0)) \cup S(J(1,1,-1)) \cup S(J(-1,-1,1)) \cup S(J(-1,0,1))$, and so $\dim S_7 = 6$.

Moreover, $S_7 \setminus (S_2 \cup S_6) = S(J(0,0,1)) \cup S(J(1,1,0)) \cup S(J(-1,0,1)) \cup S(J(0,0,-1)) \cup S(J(-1,-1,0))$,
and so $\dim S_7 \setminus (S_2 \cup S_6) = 6 > 4 = \dim(S_2 \cup S_6)$.

Example 5.7. The solution set S_8 of $X(X^2 + I) = O$ is equal to $S(J(0,0,0)) \cup S(J_c(0,0,1))$, and so $\dim S_8 = 6$.

Moreover, $S_8 \setminus (S_1 \cup S_2) = S(J_c(0,0,1))$,
and so $\dim S_8 \setminus (S_1 \cup S_2) = 6 > 0 = \dim(S_1 \cup S_2)$.

Example 5.8. The solution set S_9 of $X^3 = O$ is equal to $S(J(0,0,0)) \cup S(J_1(0,0,0)) \cup S(J_2(0,0,0))$, and so $\dim S_9 = 6$.

Moreover, $S_9 \setminus (S_2 \cup S_4) = S(J_2(0,0,0))$,
and so $\dim S_9 \setminus (S_2 \cup S_4) = 6 > 4 = \dim(S_2 \cup S_4)$.

6. The structure of S for $m \geq 4$

Let the equation (3) have distinct real roots $r_1, \ldots, r_k$ and distinct non-real roots $c_1, \ldots, c_l$. Using these two sets of numbers one can construct only finitely many non-similar real Jordan canonical forms $J_1, \ldots, J_p$ which are elements of $M_m(\mathbb{R})$.

Let S be the solution set of the corresponding equation (2). Then S is the disjoint union $\cup_{i=1}^{p} Conj_{J_i}(GL_m(\mathbb{R}))$ which is σ-compact, and $\dim S$ is -1 or the maximum of $\dim Conj_{J_i}(GL_m(\mathbb{R})), i \leq p$. In the case when the equation (3) has only distinct real roots and the number of the roots is at least m, the above maximum is achieved when J_i is a diaganol matrix with distinct roots on the diagonal. Since only diagonal matrices commute with a given diagonal matrix which has distinct diagonal entreis, Corollary 2.3 yields the following observation.

Proposition 6.1. *If* (3) *has only distinct real roots and the degree of* (3) *is greater than or equal to m, then the dimension of S is $m^2 - m$.*

If (3) does not have real roots and m is odd, then there is no real Jordan canonical forms that can be constructed to be solutions of (2), and therefore the solution set is empty. Hence, we have the following proposition.

Proposition 6.2. *If* (3) *has only non-real roots and $m = 2k + 1, k \geq 1$, then the dimension of S is -1.*

7. Concluding remarks

Question 7.1. *Let $m \geq 4$. What values can* $\dim S$ *admit?*

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